

An Analysis of Nonsimilar Problems on Spherical Blast Waves

Fumio Higashino *

Stoßwellenlabor, Institut für Luft- und Raumfahrt der RWTH, Aachen, BRD

(Z. Naturforsch. 31a, 723–727 [1976]; received April 24, 1976)

An approximate analytical method that is valid for the entire propagation regime of a blast wave is established for point explosions. The method is based on an application of the shock expansion method to boundary conditions. The numerical calculations were carried out for spherical flows with constant adiabatic exponent.

The results show that the value of the shock decay coefficient gives good agreement with the numerically exact solution by Goldstine and von Neumann. The profiles of the pressure, density, and particle velocity behind the shock were obtained. The solutions show the possible profiles as compared to other existing analyses.

1. Introduction

The theory of blast waves is a very fundamental one in gasdynamics and has been applied to various problems in hypersonic aerodynamics, astrophysics^{1, 2} and so on. In the present work, an analytical method that is valid for the entire propagation regime of a blast wave is established.

It is well known that one can get self-similarity solutions, when shock waves are very strong. The solutions which have been obtained by Taylor³, Sedov⁴, and von Neumann⁵ are individually based on dimensional analysis and independent of shock strength. On the other hand, the effect of counterpressure in front of the shock is significant for the finite strength of shock waves. In this case there is no classical self-similarity solution. However, one can get a similarity solution by assuming suitable approximations for the effect. Sakurai⁶ has accounted for the effect by expressing the asymptotic formulae of the pressure, density, and particle velocity in power series of $1/M_s^2$, respectively, where M_s is the shock Mach number. Oshima⁷ has got a quasi-similarity solution by assuming the separation of variables on the shock conditions. Bach and Lee⁸ have assumed the density profile behind the shock the exponent of which is determined from the mass conservation integral following the method by Rae⁹ and Porzel¹⁰.

Apart from the above analytical methods, numerical analysis based on the concept of artificial viscosity have been made by Goldstine and von Neumann¹¹ and Brode¹². In spite of giving exact solu-

tions, it is difficult to predict the motion of decaying shocks from their calculations for the entire regime. To provide an adequate description of the propagation of shock waves, Witham^{13, 14} has proposed a simple model. Lick¹⁵ has discussed on Whitham's rule and suggested the validity of the shock expansion method in order to solve shock trajectories for blast waves.

The main procedure of the present method is to estimate the effect of the counterpressure on propagating shock waves by making use of the shock expansion method. This enables us to apply characteristic relations for the determination of the propagating speed of shock waves and compatibility relations along the characteristics for a homentropic flow behind the shock. It should be noted that two different classes of solutions are possible depending on the direction of traversing disturbances which may be satisfied along either the positive or negative characteristic.

Numerical results show that the present analysis gives a good approximation for the motion of shock waves in the entire regime of propagation. Especially those solutions taking into account the disturbance along the negative characteristic are extremely accurate in comparison with the other existing analytical methods, especially with the exact numerical solution. In addition, the profiles of the pressure, density and particle velocity behind the shock are predictable.

2. Basic Equations and Boundary Conditions

The basic conservation equations for the adiabatic one-dimensional unsteady flow of an inviscid

* On leave from Department of Mechanical Engineering, Tokyo-Noko University, Tokyo, Japan.



and non-conducting perfect gas behind shock waves can be written as follows:

Conservation of mass:

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} + j \frac{\rho u}{x} = 0.$$

Conservation of momentum:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0. \quad (1)$$

Conservation of energy:

$$\frac{\partial e}{\partial t} + u \frac{\partial e}{\partial x} - \frac{p}{\rho^2} \left(\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} \right) = 0.$$

Here p , ρ , and u are the pressure, density, and particle velocity, respectively. x and t are the distance from the center of explosion and time, respectively. The specific internal energy of the gas, e , is given with the ratio of specific heats γ as:

$$e = \frac{1}{\gamma - 1} \frac{p}{\rho}. \quad (2)$$

The numerical constant j takes the values of 0 for planar, 1 for cylindrical, and 2 for spherical flows, respectively.

The boundary conditions across the shock front are given by the normal shock relations. They may be expressed as:

$$\begin{aligned} \rho_0 \dot{R}_s &= \rho (\dot{R}_s - u), \\ p_0 + \rho_0 \dot{R}_s^2 &= p + \rho (\dot{R}_s - u)^2, \\ \frac{p_0}{\rho_0} + e_0 + \frac{1}{2} \dot{R}_s^2 &= \frac{p}{\rho} + e + \frac{1}{2} (\dot{R}_s - u)^2 \end{aligned} \quad (3)$$

where \dot{R}_s is the shock speed and the subscript zero denotes the physical quantities in front of the shock.

Another boundary condition for point source explosions exists at the center of symmetry where the particle velocity should be zero by reason of symmetry. Then the following equation should be satisfied:

$$u(0, t) = 0. \quad (4)$$

3. The Method of Analysis

In order to evaluate the effect of the counter-pressure the shock expansion method is applied to the boundary conditions. From the system of hyper-

bolic equations, physical characteristics can be obtained¹⁶ as:

$$dx/dt = u \pm a, \quad (5)$$

with corresponding state characteristics as

$$du = \mp \frac{a}{\gamma p} dp \mp j \frac{a u}{x} dt. \quad (6)$$

Here a is the speed of sound and related to the pressure, density, and the ratio of specific heats in the flow field as:

$$a^2 = \gamma p / \rho. \quad (7)$$

As will be seen later on, the basic equations are integrated along constant time lines in the (x, t) plane. In this case from Eqs. (6) and (7) one can get the following relation as:

$$dp = \mp \rho a du, \quad d\rho = \mp (\rho/a) du. \quad (8)$$

These equations are satisfied along the fixed physical characteristics. The value of du in Eqs. (8) may differ across the characteristics on the constant time lines and depends on the shock strength at a certain time. The exact value, however, should only be decided upon after the complete calculation by the method of nonsteady characteristics. In the present work, however, disturbances due to du are taken to be constant on given lines at constant time and assumed to be equal to the value immediately behind the shock, since the change in the particle velocity with respect to time is very small. This treatment is approximately true for a wide range of shock strengths. Although the present assumption makes analysis easy, secondary shock waves cannot be predicted, since the flow field is treated as a simple wave region. If one can evaluate the change in the particle velocity more precisely on the given line, the formation of the secondary shock wave may also be predicted. Anyway, with the above approximation one can estimate changes in the pressure and density at distances from the origin.

4. Transformation of Variables

For convenience we transform the independent variables (x, t) into nondimensional variables (ξ, M_s) as follows:

$$\xi = \frac{x}{R_s}, \quad M_s = \frac{\dot{R}_s}{a_0}, \quad \frac{dR_s}{dt} = \dot{R}_s, \quad (9)$$

where R_s is the distance of the shock front from the center and M_s the shock Mach number.

The dependent variables of the flow, the pressure p , density ρ , and particle velocity u are also transformed into the nondimensional forms f , g and h , respectively. They are given as

$$\begin{aligned} p(x, t) &= \rho_0 \dot{R}_s^2 f(\xi, M_s), \\ \rho(x, t) &= \rho_0 g(\xi, M_s), \\ u(x, t) &= \dot{R}_s h(\xi, M_s). \end{aligned} \quad (10)$$

The basic Eqs. (1) can be written with the nondimensional variables as ¹⁷:

$$\begin{aligned} (h - \xi) \frac{\partial g}{\partial \xi} + g \frac{\partial h}{\partial \xi} + j \frac{g h}{\xi} + \theta M_s \frac{\partial g}{\partial M_s} &= 0, \\ (h - \xi) \frac{\partial h}{\partial \xi} + \frac{1}{g} \frac{\partial f}{\partial \xi} + h \theta + \theta M_s \frac{\partial h}{\partial M_s} &= 0, \\ (h - \xi) \left(\frac{\partial f}{\partial \xi} - \gamma \frac{f}{g} \frac{\partial g}{\partial \xi} \right) \\ + 2 f \theta + \theta M_s \left(\frac{\partial f}{\partial M_s} - \gamma \frac{f}{g} \frac{\partial g}{\partial M_s} \right) &= 0, \end{aligned} \quad (11)$$

where θ is defined as:

$$\theta = \frac{R_s}{M_s} \frac{dM_s}{dR_s} \quad (12)$$

and called decay coefficient.

For the boundary conditions across the shock front the Rankine-Hugoniot relations, which give the conditions immediately behind the shock front at $\xi = 1$, can be given from Eqs. (3) as follows:

$$\begin{aligned} f(1) &= \frac{2}{\gamma + 1} - \frac{\gamma - 1}{\gamma(\gamma + 1)M_s^2}, \\ g(1) &= \frac{(\gamma + 1)M_s^2}{(\gamma - 1)M_s^2 + 2}, \\ h(1) &= \frac{2}{\gamma + 1} \left(1 - \frac{1}{M_s^2} \right). \end{aligned} \quad (13)$$

The boundary condition at the center $\xi = 0$, is expressed from Eq. (9) as:

$$h(0) = 0. \quad (14)$$

Now to express the derivatives with respect to shock Mach numbers in Eqs. (11) explicitly, the characteristic relations of Eqs. (5), (7) and (8) are utilized. Substitution of Eqs. (9) into (5) yields

$$\dot{R}_s = u \pm a \quad (15)$$

on the shock front where $\xi = 1$. The total derivative of (15) gives

$$d\dot{R}_s = \frac{\gamma + 1}{2} du. \quad (16)$$

Here the relation of a homentropic flow is used. The above equation determines the variation of the particle speed due to that of the shock speed. For a given shock Mach number Eq. (16) can be given in nondimensional form as:

$$M_s \frac{\partial h}{\partial M_s} = \frac{2}{(\gamma + 1)M_s^2}. \quad (17)$$

The variation of the pressure and density of the flow behind the shock can be obtained from Eqs. (8) with (17)

$$\begin{aligned} M_s \frac{\partial g}{\partial M_s} &= \mp \frac{2g}{(\gamma + 1)M_s^2} \sqrt{\frac{g}{\gamma f}}, \\ M_s \frac{\partial f}{\partial M_s} &= \mp \frac{2\sqrt{\gamma f g}}{(\gamma + 1)M_s^2}. \end{aligned} \quad (18)$$

The above equations with the positive sign denote disturbances which propagate against the flow. This implies the application of the shock expansion method to the boundary conditions, while the use of the equations with the negative sign corresponds to the application of Whitham's rule. By substituting Eqs. (17) and (18) into Eqs. (11), the basic equations are reduced to a set of ordinary differential equations with respect to the distance ξ only. Especially in the case of very strong shock waves the right hand sides of Eqs. (17) and (18) vanish, since the terms with $1/M_s^2$ tend to naught. Thus this set of basic equations fulfills the classical self-similarity solution. The equations can easily be integrated by usual numerical methods.

It should be noted that the energy equation in the basic equations is very much simplified under the present approximation. On substituting Eqs. (18) into the third equation in Eqs. (11), the derivatives with respect to the shock Mach number M_s are cancelled.

5. Numerical Calculations

The set of the ordinary differential equations with the boundary conditions was integrated by the Runge-Kutta-Gill method. The integration was carried out from $\xi = 1$ at the shock front with Eqs. (14) to $\xi = 0$ at the center. To determine the correct

value of the decay coefficient θ which may approximately seem to satisfy the condition of Eq. (15), the iteration method was used.

In all calculations the step size used here is $\Delta\xi = 0.0025$ and the required accuracy for the decay coefficient is less than six places of decimals. Numerical calculations were performed both for positive and negative signs in Eqs. (18) and covered a wide range of shock Mach numbers. As a typical example, spherical blast waves in gases of diatomic molecules with $\gamma = 1.4$ are examined. The physical quantities are normalized by the values at the shock front.

6. Results and Discussions

The change in the decay coefficient with shock strength is shown in Figure 1. In the Figure, I and II correspond to the positive and negative sign in Eq. (18), respectively, and $\eta = 1/M_s^2$; the self-similarity solution is shown as the case of $\eta = 0$. The result for the negative sign shows that the strength of the shock wave which propagates outside the center is rapidly weakened in comparison with that for the positive one. This is a consequence of overestimation regarding the disturbances on the constant time line. On the other hand, the use of the negative sign corresponds to an approximation which takes into account reflected disturbances from the shock front only. As has been pointed out by Lick¹⁵, this approximation may give good results for the estimation of the propagating shock speed and the flow field behind the shock. In spite of the comparatively simple approximation, the calculated

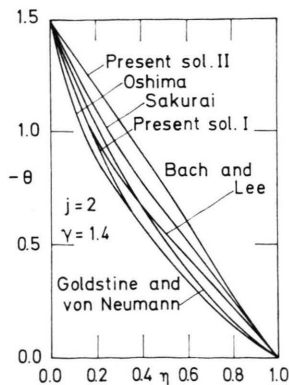


Fig. 1. The variation of the decay coefficient with shock strength.

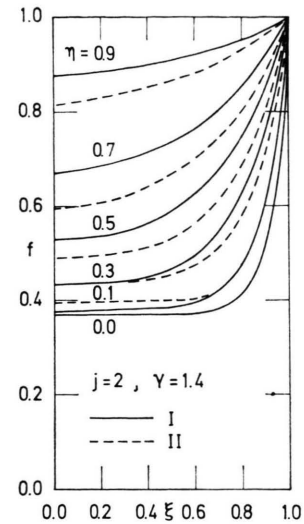


Fig. 2. The profiles of nondimensional pressure behind shock waves.

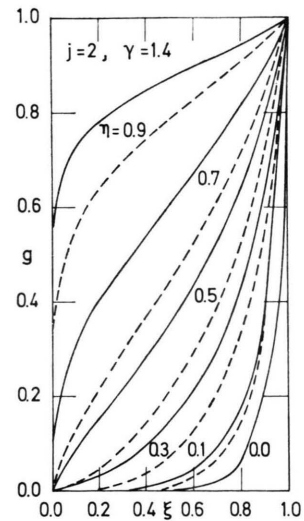


Fig. 3. The profiles of nondimensional density behind shock waves.

values of the decay coefficient for the case of the positive sign in Eq. (18) give quite accurate solutions for the entire propagation regime of a blast wave, when compared with the exact numerical solution by Goldstine and von Neumann¹¹. This may be explained by the fact that the change in the particle velocity due to the variation of the propagating shock strength is very small and the distribution of the density behind the shock plays an important role to the nonsimilarity⁸.

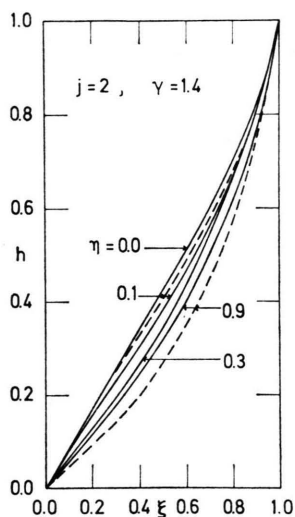


Fig. 4. The profiles of nondimensional particle velocity behind shock waves.

The profiles of the nondimensional pressure f , density g , and particle velocity h behind the shock front are shown in Figs. 2, 3 and 4, respectively. In these figures the solid and dotted lines correspond to the results using the positive and negative sign in Eq. (18), respectively. Differences between two profiles at the same shock strength are remarkable for weak shock waves. However, one can find that the profiles are very similar and close to each other without regard to the sign in Equations (18). In addition to the above fact, the slope of the particle velocity is nearly constant behind the shock front. Therefore the present approximation is justified for

point explosions. This solution differs from the one reported by Bach and Lee⁸. The particle velocity should take a single value for a given shock Mach number of the blast wave. Other profiles for the pressure and density show relatively good agreement with their solutions.

7. Conclusions

To solve the problem on blast waves, a simple method has been proposed here. The method is based on the application of the shock expansion method to boundary conditions. It should be noted that two classes of solutions are possible. In spite of making a simple approximation, the solutions showed fairly good agreement for the decay coefficient compared with the exact numerical analysis by Goldstine and von Neumann¹¹, when the disturbances with the positive sign in Eq. (18) are assumed. Furthermore, the present analytical method is valid for the entire propagation regime of a blast wave. The profiles for the pressure, density and particle velocity behind the shock front are reliable.

Acknowledgement

The author would like to express his appreciation to Prof. H. Grönig, RWTH Aachen, for helpful discussions and interest. He is also indebted to Mr. Klose and Mrs. Eckert for considerable assistance in preparation.

This work was accomplished during a stay of the author in the Stoßwellenlabor der RWTH as a research fellow. He wishes to acknowledge his grateful thanks to the Alexander von Humboldt-Stiftung.

¹ W. D. Hayes and R. Probstein, *Hypersonic Flow Theory*, Vol. 1, Academic Press, New York 1966, p. 54.

² Ya. B. Zeldovich and Yu. P. Rayzer, *Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena*, Edited by W. D. Hayes and R. F. Probstein, Academic Press, New York 1966, p. 916.

³ G. I. Taylor, *Proc. Roy. Soc. London A* **201**, 175 [1950].

⁴ L. I. Sedov, *Similarity and Dimensional Methods in Mechanics*, Inforsearch Ltd., London 1959, p. 146.

⁵ J. von Neumann, *Collected Works of J. von Neumann*, Vol. 4, Pergamon Press, London 1963, p. 176.

⁶ A. Sakurai, *Basic Developments in Fluid Dynamics*, Edited by M. Holt, Academic Press, London 1965, p. 309.

⁷ K. Oshima, *Aero. Res. Inst., Univ. Tokyo, Rep. No. 36* [1964].

⁸ G. G. Bach and J. H. S. Lee, *Amer. Inst. Aero. Astr. J.* **8**, 271 [1970].

⁹ W. J. Rae, NASA CR-54251 [1965].

¹⁰ F. B. Porzel, Los Alamos Scientific Lab., Rep. LA-1664 [1954].

¹¹ H. Goldstine and J. von Neumann, *Collected Works of J. von Neumann*, Vol. 4, Pergamon Press, London 1963, p. 386.

¹² H. Brode, *J. Appl. Phys.* **26**, 766 [1955].

¹³ G. B. Whitham, *Proc. Roy. Soc., London A* **203**, 571 [1950].

¹⁴ G. B. Whitham, *J. Fluid Mech.* **4**, 337 [1958].

¹⁵ W. Lick, *J. Fluid Mech.* **25**, 179 [1966].

¹⁶ P. C. Chou, R. R. Karpp, and S. L. Huang, *AIAA J.* **5**, 618 [1967].

¹⁷ F. Higashino, *J. Facult. Eng., Univ. Tokyo (B)* **32**, 105 [1973].